插值法 Interpolation

意義

尋找穿過所有給定點的函數。

公式

1. 牛頓插值法

$$f(\mathrm{x}) = \mathrm{a}(\mathrm{x} - x_1)(x - x_2) + b(x - x_1) + c$$

2. 拉格朗日插值法

$$f(\mathrm{x}) = f(x_1)\frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + f(x_2)\frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + f(x_3)\frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

3. 范德蒙矩陣

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

範例

求滿足 f(2)=4、f(3)=1、f(4)=2 的多項式，也就是同時通過(2,4)、(3,1)、(4,2)三個點的函數。

牛頓插值法 Newton Interpolation

$f(2) = 4$

$f(3) = 4 + \alpha(\mathrm{x} - 2) = 1$

$4 + \alpha(3 - 2) = 1$

$\alpha = -3$

$f(4) = f(3) + \beta(\mathrm{x} - 2)(\mathrm{x} - 3) = 2$

$4 + (-3)(4 - 2) + \beta(4 - 2)(4 - 3) = 2$

$\beta = 2$

$f(\mathrm{x}) = 4 - 3(\mathrm{x} - 2) + 2(x - 2)(x - 3)$

$= 2x^2 - 13x + 22$

拉格朗日插值法 Lagrange Interpolation

	k1(x)	k2(x)	k3(x)	f(x)
x=2	4	0	0	4
x=3	0	1	0	1
X=4	0	0	2	2

$$f(x) = 4 \times \frac{(x-3)(x-4)}{(2-3)(2-4)} + 1 \times \frac{(x-2)(x-4)}{(3-2)(3-4)} + 2 \times \frac{(x-2)(x-3)}{(4-2)(4-3)}$$
$$= 4 \times \frac{(x-3)(x-4)}{2} + 1 \times \frac{(x-2)(x-4)}{-1} + 2 \times \frac{(x-2)(x-3)}{2}$$
$$= 2(x-3)(x-4) - (x-2)(x-4) + (x-2)(x-3)$$
$$= 2(x^2 - 7x + 12) - (x^2 - 6x + 8) + (x^2 - 5x + 6)$$
$$= 2x^2 - 13x + 22$$

范德蒙矩陣 Vandermonde method

設通過三個點的二次內插多項式為$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2$，滿足

$f(2) = \beta_0 + \beta_1 2 + \beta_2 2^2 = 4$

$f(3) = \beta_0 + \beta_1 3 + \beta_2 3^2 = 1$

$f(4) = \beta_0 + \beta_1 4 + \beta_2 4^2 = 2$

范德蒙矩陣為

$$\begin{bmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^3 \\ 1 & 4 & 4^2 \end{bmatrix} \times \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^3 \\ 1 & 4 & 4^2 \end{bmatrix}^{-1} \times \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 22 \\ -13 \\ 2 \end{bmatrix}$$

方程式聯立解

設內插多項式$y = ax^2 + bx + c$，將(2,4)、(3,1)、(4,2)代入

$$\begin{cases} 2^2 a + 2b + c = 4 \\ 3^2 a + 3b + c = 1 \\ 4^2 a + 4b + c = 2 \end{cases} \rightarrow \begin{cases} 4a + 2b + c = 4 \\ 9a + 3b + c = 1 \\ 16a + 4b + c = 2 \end{cases}$$

令$c = 4 - 4a - 2b$，代入

$$\begin{cases} 9a + 3b + 4 - 4a - 2b = 1 \\ 16a + 4b + 4 - 4a - 2b = 2 \end{cases} \rightarrow \begin{cases} 5a + b = -3 \\ 12a + 2b = -2 \end{cases} \rightarrow \begin{cases} -10a - 2b = 6 \\ 12a + 2b = -2 \end{cases}$$

$2a = 4$，$a = 2$

$-2b = 26$，$b = -13$

$c = 4 - 4(2) - 2(-13) = 22$